Quantization of semi-classical twists and noncommutative geometry

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Abstract

A problem of defining the quantum analogues for semi-classical twists in $U(\mathfrak{g})[[t]]$ is considered. First, we study specialization at $q = 1$ of singular coboundary twists defined in $U_q(\mathfrak{g})[[t]]$ for $\mathfrak{g}$ being a nonexceptional Lie algebra, then we consider specialization of noncoboundary twists when $\mathfrak{g} = \mathfrak{sl}_3$ and obtain $q-$deformation of the semi-classical twist introduced by Connes and Moscovici in noncommutative geometry.

Keywords: Noncommutative geometry, Hopf algebras

1 Introduction

Hopf algebras play an increasingly important role in noncomutative geometry and Quantum Field Theory. One of the sources for producing new types of Hopf algebras is twisting, a deformation of the coalgebraic structure of a given Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$ preserving the algebraic structure (H, μ, η). Such deformations are generated by the twisting elements (twists) $\mathcal{F} \in (H \otimes H)$ satisfying the conditions

$$\begin{aligned} \mathcal{F}^{12}(\Delta \otimes \mathrm{id})(\mathcal{F}) &= \mathcal{F}^{23}(\mathrm{id} \otimes \Delta)(\mathcal{F}) \\ (\varepsilon \otimes \mathrm{id})(\mathcal{F}) &= (\mathrm{id} \otimes \varepsilon)(\mathcal{F}) = 1 \end{aligned} \tag{1}$$

that guarantee $H^{\mathcal{F}} \equiv (H, \mu, \eta, \mathrm{Ad}\mathcal{F} \circ \Delta, \epsilon, S)$ is a new Hopf algebra. In fact, when H is not finite dimensional, $\mathcal{F}$ is usually defined in some completion of the tensor product and H is understood to be a topological Hopf algebra.

In this article we consider two types of twists: the semi-classical ones if $H = U(\mathfrak{g})[[t]]$ and the quantum ones if $H = U_q(\mathfrak{g})[[t]]$. Some of the semi-classical deformations such as those defined by the Jordanian twists [5, 10] appear as the limiting cases of the quantum ones (in the sense that specialization

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at $q = 1$ is extended to work for topological Hopf algebras). It was a motivation for us to study the quantum twists as many computational problems involved into a direct check of (1) drastically resolve when one works with $U_q(\mathfrak{g})[[t]]$ instead of $U(\mathfrak{g})[[t]]$ and thus the quantum twists is a source for many universal deformation formulas in the sense of [5].

The work is organized as follows. After preliminary section intended to fix notations, we show that if $\mathfrak{g}$ is a nonexceptional simple Lie algebra, then a quantum analogue of the Jordanian twist can be taken to be a coboundary twist in $U_q(\mathfrak{g})[[t]]$:

$$\mathcal{J}(e_\lambda) := (W \otimes W)\Delta(W^{-1}), \text{ where } W = \exp_{q_\lambda}(\frac{t}{1-q_\lambda}\ e_\lambda); \qquad q_\lambda := q^{(\lambda,\lambda)}$$

with e_λ being a quantum highest root generator in some quantum Cartan-Weyl basis. We prove that $\mathcal{J}(e_\lambda)$ is nonsingular and specializes to a nontrivial twisting of $U(\mathfrak{g})[[t]]$. As an application of the Jordanian twists [5, 10] to noncommutative geometry [1], we prove that there is a homomorphism of the Connes-Moscovici Hopf algebra $\iota : \mathcal{H}_1 \to U^{\mathcal{F}}(\mathfrak{sl}_3)[[t]]$, with $\mathcal{F}$ being a Jordanian twist, where $\mathcal{H}_1$ has the following structure

$$\begin{gathered}[Y, X] = X, \qquad [Y, \delta_n] = n\delta_n, \qquad [X, \delta_n] = \delta_{n+1}, \qquad [\delta_k, \delta_l] = 0, \qquad k, l \geq 1 \\ \Delta(Y) = Y \otimes 1 + 1 \otimes Y, \qquad \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1 \\ \Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y.\end{gathered} \tag{2}$$

Through factoring $\mathcal{H}'_1 := \mathcal{H}_1 / < \delta_2 - \frac{1}{2}\ \delta_1^2 >$, one obtains in fact an embedding

$$\iota : \mathcal{H}'_1 \hookrightarrow U^{\mathcal{F}}(\mathfrak{sl}_3)[[t]]$$

and the twist found in [1]:

$$F = \sum_{n\geq 0} t^n \sum_{k=0}^{n} \frac{S(X)^k}{k!}(2Y + k)_{n-k} \otimes \frac{X^{n-k}}{(n-k)!}(2Y + n - k)_k \tag{3}$$

where $(\alpha)_k = \alpha(\alpha + 1)\cdots(\alpha + k - 1)$ and $S(X) = -X + \delta_1 Y$, can be obtained as a pullback $F = \iota_* \Phi$ of a semi-classical twist Φ in $U^{\mathcal{F}}(\mathfrak{sl}_3)[[t]]$. In section **??** we show that ι can be "quantized", thus leading to a quantum analogue of $\mathcal{H}'_1$:

$$kxk^{-1} = q^2\ x, \qquad kzk^{-1} = q^2\ z, \qquad q^2 xz - zx = -tz^2$$

$$\begin{gathered}\Delta(k) = k \otimes k, \qquad \Delta(z) = z \otimes k + 1 \otimes z \\ \Delta(x) = x \otimes k^{-1} + 1 \otimes x + t\ z \otimes \frac{(k - k^{-1})}{1 - q^2}.\end{gathered}$$

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2 Preliminarities

Let $\mathfrak{g}$ be a simple Lie algebra with the set of simple roots $\pi = \{\alpha_1, \dots, \alpha_N\}$ and the Cartan matrix $(A)_{ij} = a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. By definition, a Hopf algebra $U_q(\mathfrak{g})$ is generated by $\{e_i, f_i, k_i^{\pm 1}\}_{1 \le i \le N}$ over $\mathbb{C}(q)$ which are subject to the following relations

$$k_i e_j k_i^{-1} = q^{(\alpha_i, \alpha_j)} e_j, \qquad k_i f_j k_i^{-1} = q^{-(\alpha_i, \alpha_j)} f_j \tag{4}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \tag{5}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i} (e_i)^n e_j (e_i)^{1-a_{ij}-n} = 0 \ \text{ for } i \neq j \tag{6}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i} (f_i)^n f_j (f_i)^{1-a_{ij}-n} = 0 \ \text{ for } i \neq j \tag{7}$$

where $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$ and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q \equiv \frac{[m]_q!}{[n]_q! [m-n]_q!}$$

$$[k]_q! \equiv [1]_q [2]_q \dots [k]_q, \quad [l]_q \equiv (q^l - q^{-l})/(q - q^{-1})$$

The Hopf algebra structure is defined uniquely by fixing the values of the co-product on the Chevalley generators

$$\Delta(k_i) = k_i \otimes k_i \tag{8}$$

$$\Delta(e_i) = k_i^{-1} \otimes e_i + e_i \otimes 1, \quad \Delta(f_i) = f_i \otimes k_i + 1 \otimes f_i \tag{9}$$

$$S(k_i) = k_i^{-1}, \quad S(e_i) = -k_i e_i, \quad S(f_i) = -f_i k_i^{-1} \tag{10}$$

$$\varepsilon(k_i) = 1, \qquad \varepsilon(e_i) = 0, \qquad \varepsilon(f_i) = 0. \tag{11}$$

Letting $q_h = e^h$ and $K_i := q_h^{h_i}$ in (4)-(11), we come to definition of $U_h(\mathfrak{g})[[h]]$, the topological Hopf algebra over $\mathbb{C}[[h]]$.

One introduces a linear ordering on the set of positive roots Δ_+ by fixing the reduced decomposition of the longest element in the Weyl group $w_0 = s_{i_1} s_{i_2} \cdots s_{i_M}$. Then the linear ordering read from the left to the right is the following

$$\Delta_+ = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \dots, s_{i_1} s_{i_2} \cdots s_{i_{M-1}} \alpha_{i_M}\}. \tag{12}$$

An ordering (12) is normal, namely for each $\alpha, \beta \in \Delta_+$ such that $\alpha + \beta \in \Delta_+$ and $\alpha \prec \beta$, we have $\alpha \prec \alpha + \beta \prec \beta$. There is one-to-one correspondence between the reduced decompositions of the longest element in the Weyl group and the normal orderings given by (12). Following [8], one defines the generators corresponding to the composite roots. For a chosen normal ordering on Δ_+ let $\alpha, \beta, \gamma \in \Delta_+$

be pairwise noncollinear roots, such that $\gamma = \alpha + \beta$. Let α and β are taken so that there are no other roots α' and β' with the property $\gamma = \alpha' + \beta'$. Then if $e_{\pm\alpha}$ and $e_{\pm\beta}$ have already been constructed, we set

$$e_\gamma = e_\alpha e_\beta - q^{-(\alpha,\beta)} e_\beta e_\alpha, \quad e_{-\gamma} = e_{-\beta} e_{-\alpha} - q^{-(\beta,\alpha)} e_{-\alpha} e_{-\beta}.$$

For any root $\gamma \in \Delta_+$ define

$$\check{R}_\gamma := \exp_{q_\gamma}(-(q - q^{-1}) a_\gamma^{-1}\ e_\gamma k_\gamma \otimes k_\gamma^{-1} f_\gamma),$$

where $q_\gamma = q^{(\gamma,\gamma)}$ and

$$\exp_q(x) := \sum_{n \geq 0} \frac{x^n}{(n)_q!}, \quad (n)_q! \equiv (1)_q (2)_q \ldots (n)_q, \quad (k)_q \equiv (1 - q^k)/(1 - q).$$

with factors a_γ coming from the relations

$$[e_\gamma, e_{-\gamma}] = a_\gamma \frac{k_\gamma - k_\gamma^{-1}}{q - q^{-1}}.$$

The elements $\check{R}_\gamma$ are understood to be taken in some completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ (the Taylor extension or the $h-$adic [8]). Now the coproducts of composite root generators can be expressed in terms of the adjoint action of the following factors

$$\check{R}_{\prec\beta} := \prod_{\gamma \prec \beta} \check{R}_\gamma,$$

where the product over all the positive roots such that $\prec \beta$ is taken in accordance with the chosen normal ordering. Namely, we have the following

Theorem 1 ([8, 9]). *Consider the canonical isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ defined by the bilinear form $(\ ,\ )$ on $\mathfrak{h}$. Let $h_\beta \in \mathfrak{h}$ be the image of a root $\beta \in \mathfrak{h}^*$ with respect to this isomorphism. Then the following identity holds:*

$$\Delta(e_\beta) = \check{R}_{\prec\beta}(k_\beta^{-1} \otimes e_\beta + e_\beta \otimes 1)\check{R}_{\prec\beta}^{-1}. \tag{13}$$

Proof. Note that (13) is equivalent to Proposition 8.3 from [8] if one applies $(S \otimes S)$ to both parts of

$$\Delta^{op}(e_\alpha) = \left(\prod_{\gamma<\alpha} \check{R}_\gamma\right)(1 \otimes e_\alpha + e_\alpha \otimes \overline{k}_\alpha)\left(\prod_{\gamma<\alpha} \check{R}_\gamma\right)^{-1}$$

(notations are of [8]) and check that our construction of the modified basis differs by the change $q \leftrightarrow q^{-1}$. $\square$

We conclude this section by adding remarks on specialization [3]. Let $\mathcal{A} = \mathbb{C}[q, q^{-1}]_{(q-1)}$ be a ring of rational functions regular at $q = 1$ and let $\hat{U}_q(\mathfrak{g})$ be an $\mathcal{A}$ subalgebra in $U_q(\mathfrak{g})$ generated by $\left\{e_i, f_i, k_i^{-1}, \dfrac{k_i - 1}{q - 1}\right\}$ then

$$\hat{U}_q(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{C} \approx U(\mathfrak{g})$$

where $\mathbb{C}$ is regarded as an $\mathcal{A}$ module (q acts as 1). By construction of the quantum Cartan-Weyl basis we have the property

$$\Delta(e_\beta) \in \hat{U}_q(\mathfrak{g}) \otimes \hat{U}_q(\mathfrak{g})$$

but, in fact, one can deduce from (13) more restrictive property

$$\Delta(e_\beta) - k_\beta^{-1} \otimes e_\beta - e_\beta \otimes 1 \in (q - q^{-1})\ \hat{U}_q^+(\mathfrak{g}) \otimes \hat{U}_q^+(\mathfrak{g}) \tag{14}$$

where $\hat{U}_q^+(\mathfrak{g})$ is generated by $\left\{ e_i, k_i^{-1}, \dfrac{k_i - 1}{q - 1} \right\}$. In what follows we are usually working rather with completions $\hat{U}_q(\mathfrak{g})[[t]]$ and $\hat{U}_q(\mathfrak{g}) \otimes \hat{U}_q(\mathfrak{g})[[t]]$ in which the twists are to be defined. Let us formulate the following simple result which is of value for further application

Proposition 1. *Let $\mathcal{F} \in \hat{U}_q(\mathfrak{g}) \otimes \hat{U}_q(\mathfrak{g})[[t]]$ be a twist in $U_q(\mathfrak{g})[[t]]$, then its specialization $\overline{\mathcal{F}}$, obtained by order-wise specialization of its coefficients from $\hat{U}_q(\mathfrak{g}) \otimes \hat{U}_q(\mathfrak{g})$ at $q = 1$, is still a twist in $U(\mathfrak{g})[[t]]$.*

Proof. Indeed,representing $\mathcal{F}$ as a series

$$\mathcal{F} = 1 \otimes 1 + \mathcal{F}_1 t + \mathcal{F}_2 t^2 + \cdots$$

we see that (1) is a equivalent to an infinite set of identities and each of them after specializing $q = 1$ remain valid. Thus

$$\overline{\mathcal{F}} = 1 \otimes 1 + \overline{\mathcal{F}}_1 t + \overline{\mathcal{F}}_2 t^2 + \cdots$$

is a twist in $U(\mathfrak{g})[[t]]$. $\square$

3 Quantum Jordanian twists

We restrict ourselves to consideration of nonexceptional Lie algebra $\mathfrak{g}$ and define the quantum Jordanian twists as those specializing to semi-classical ones which define quantization of skew-symmetric extended Jordanian $r-$matrices:

$$r_\lambda = H_\lambda \wedge E_\lambda + 2 \sum_{\gamma_1 \prec \gamma_2, \gamma_1 + \gamma_2 = \lambda} E_{\gamma_1} \wedge E_{\gamma_2}$$

by the rule

$$\mathcal{R} = \mathcal{F}_{\lambda 21} \mathcal{F}_\lambda^{-1} = 1 \otimes 1 + t\ r_\lambda \bmod t^2,$$

where we have denoted by H_λ, E_γ the elements of the classical Cartan-Weyl basis.

Let us fix some normal ordering on Δ_+ and define a generator $e_\lambda \in U_q(\mathfrak{g})$ corresponding to the highest root λ according to the recipe from the previous section. Then nonexceptional root systems are remarkable by the following property:

Proposition 2. *Let $\mathfrak{g}$ be a non exceptional Lie algebra, then there is such a normal ordering "$\prec$" on Δ_+ so that*

$$[e_\gamma, e_\lambda]_{q^{-(\gamma,\lambda)}} = 0 \text{ for any } \gamma \prec \lambda, \text{ and } e_\gamma, e_\lambda \in U_q(\mathfrak{g}).$$

Proof. The proof is based on the expansion [8, 9]:

$$e_\gamma e_\lambda - q^{-(\gamma,\lambda)} e_\lambda e_\gamma = \sum_{\gamma \prec \gamma_1 \prec \cdots \prec \gamma_j \prec \lambda} c_{l,\gamma,\lambda} e_{\gamma_1}^{l_1} \cdots e_{\gamma_j}^{l_j} \tag{15}$$

where $c_{l,\gamma,\lambda} \in \mathbb{C}[q, q^{-1}]$. Non zero terms in the sum are subject to condition

$$\gamma + \lambda = l_1\gamma_1 + l_2\gamma_2 + \cdots + l_j\gamma_j. \tag{16}$$

In A_N we choose a normal ordering as

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \alpha_2 + \cdots + \alpha_N \succ \overbrace{\alpha_2 \succ \cdots \succ \alpha_{N-1} + \alpha_N \succ \alpha_N}^{\text{roots without } \alpha_1}$$

and $\lambda = \alpha_1 + \cdots \alpha_N$. If $\lambda \succ \gamma \succ \alpha_N$ we can satisfy (15) only with zero coefficients.

In B_N we have

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-1} + 2\beta \succ \overbrace{\alpha_2 \succ \alpha_2 + \alpha_3 \succ \cdots \succ \beta}^{\text{roots without } \alpha_1}$$

and $\lambda = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-1} + 2\beta$.

In C_N we fix the following ordering

$$\overbrace{\alpha_1 \prec \alpha_1 + \alpha_2 \prec \cdots \prec \alpha_1 + \cdots + \alpha_{N-1}}^{\text{roots without } \beta} \prec 2(\alpha_1 + \cdots + \alpha_{N-1}) + \beta \prec$$
$$\alpha_1 + \cdots + \alpha_{N-1} + \beta \prec \cdots \prec \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{N-1}) + \beta \prec \cdots \prec \alpha_2 \prec \cdots \prec \beta,$$

and $\lambda = 2(\alpha_1 + \cdots + \alpha_{N-1}) + \beta$. This ordering eliminates all non zero terms on the r.h.s of (15).

In D_N we have quite a similar situation

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \cdots + \alpha_{N-1} \succ \alpha_1 + \cdots + \alpha_{N-1} + \beta \succ$$
$$\alpha_1 + \alpha_2 + \cdots + 2\alpha_{N-2} + \alpha_{N-1} + \beta \succ \cdots \succ \alpha_1 + 2\alpha_2 + \cdots 2\alpha_{N-2} + \alpha_{N-1} + \beta \succ$$
$$\overbrace{\alpha_2 \succ \alpha_2 + \alpha_3 \succ \cdots \succ \beta}^{\text{roots without } \alpha_1},$$

and $\lambda = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{N-2} + \alpha_{N-1} + \beta$. $\square$

As a direct consequence of Proposition 2, we obtain

Proposition 3. *A $q-$commutation holds*

$$(e_\lambda \otimes 1)(\Delta(e_\lambda) - e_\lambda \otimes 1) = q_\lambda(\Delta(e_\lambda) - e_\lambda \otimes 1)(e_\lambda \otimes 1) \text{ where } q_\lambda = q^{(\lambda,\lambda)}.$$

Proof. The statement follows from Proposition 2 and (13) if one notices that

$$\check{R}_{\prec\lambda}(e_\lambda \otimes 1) = (e_\lambda \otimes 1)\check{R}_{\prec\lambda}.$$

□

Let us define a coboundary twist

$$\mathcal{J}(e_\lambda) = (W \otimes W)\Delta(W^{-1}), \text{ where } W = \exp_{q_\lambda}(\frac{t}{1-q_\lambda}\ e_\lambda)$$

Then the following is true

Proposition 4. *$\mathcal{J}(e_\lambda)$ is nonsingular and defines a nontrivial twisting of $U(\mathfrak{g})[[t]]$ in the limit $q \to 1$.*

Proof. By Proposition 3 we have

$$\mathcal{J}(e_\lambda) = \exp_{q_\lambda}(\frac{t}{1-q_\lambda}\ 1 \otimes e_\lambda)\exp_{q_\lambda^{-1}}(-\frac{t}{1-q_\lambda}\ (\Delta(e_\lambda) - e_\lambda \otimes 1)).$$

The latter representation is nonsingular and from $\hat{U}_q(\mathfrak{g}) \otimes \hat{U}_q(\mathfrak{g})[[t]$, which is obvious if one uses the Campbell-Hausdorff formula after applying the dilogarithmic representation of $q-$exponent [4]:

$$\exp_{q_\lambda}(\frac{t}{1-q_\lambda}\ x) = \exp\left(\sum_{n\geq 1} \frac{t^n}{n(1-q_\lambda^n)}\ x^n\right) \tag{17}$$

along with the properties (14) and

$$[e_\lambda, \hat{U}_q^+(\mathfrak{g})] \in (q-1)\ \hat{U}_q^+(\mathfrak{g}).$$

Note, that to be self-consistent one can directly verify that (17) satisfies (20), considering (20) as a functional equation for the function $\mathrm{Li}_2(t \cdot x, q)$:
$\mathrm{Li}_2(t \cdot x, q) = \ln(\exp_q(\frac{t}{1-q}\ x))$. Finally, $\overline{\mathcal{J}}$ is a twist by Proposition 1. □

Example 1. Consider $\mathfrak{g} = \mathfrak{sl}_{N+1}$. We have the following formula for the coproduct associated with the chosen normal ordering

$$\alpha_1 \succ \alpha_1 + \alpha_2 \succ \cdots \succ \alpha_1 + \alpha_2 + \cdots + \alpha_N \succ \overbrace{\alpha_2 \succ \cdots \succ \alpha_{N-1} + \alpha_N \succ \alpha_N}^{\text{roots without } \alpha_1}$$

given by

$$\Delta(e_{\epsilon_1-\epsilon_{N+1}}) = k^{-1}_{\epsilon_1-\epsilon_{N+1}} \otimes e_{\epsilon_1-\epsilon_{N+1}} + e_{\epsilon_1-\epsilon_{N+1}} \otimes 1 + (1-q^2) \sum_{i=1}^{N-1} e_{\epsilon_1-\epsilon_{i+1}} k^{-1}_{\epsilon_{i+1}-\epsilon_{N+1}} \otimes e_{\epsilon_{i+1}-\epsilon_{N+1}}$$

where $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

To calculate specialization $\overline{\mathcal{J}(e_{\epsilon_1-\epsilon_{N+1}})}$, we represent $\mathcal{J}(e_{\epsilon_1-\epsilon_{N+1}})$ in the following form

$$\mathcal{J}(e_{\epsilon_1-\epsilon_{N+1}}) = \exp_{q^{-2}}(-t\sum_{i=1}^{N-1} e_{\epsilon_1-\epsilon_{i+1}} k^{-1}_{\epsilon_{i+1}-\epsilon_{N+1}} \otimes e_{\epsilon_{i+1}-\epsilon_{N+1}} C_{1,N+1})\mathcal{J}_1$$

where

$$C_{1,N+1} = \exp_{q^2}(-\frac{q\ t}{1-q^2}\ e_{\epsilon_1-\epsilon_{N+1}}) \cdot \exp_{q^{-2}}(\frac{t}{1-q^2}\ e_{\epsilon_1-\epsilon_{N+1}}) \tag{18}$$

and

$$\mathcal{J}_1 = \exp_{q^2}(\frac{t}{1-q^2}\ 1 \otimes e_{\epsilon_1-\epsilon_{N+1}}) \exp_{q^{-2}}(-\frac{t}{1-q^2}\ k^{-1}_{\epsilon_1-\epsilon_{N+1}} \otimes e_{\epsilon_1-\epsilon_{N+1}}). \tag{19}$$

Calculation of (18) − (19) is based on the Heine's formula from [6]:

$$1 + \sum_{n\geq 1} t^n \frac{(\alpha)_q \cdots (\alpha+n-1)_q}{(n)_q!}\ x^n = \exp_q(\frac{t}{1-q}\ x) \exp_{q^{-1}}(-\frac{q^\alpha t}{1-q}\ x). \tag{20}$$

Note that (20) can be recast so that to hold in $\hat{U}_q(\mathfrak{g}) \otimes \hat{U}_q(\mathfrak{g})[[t]]$:

$$1\otimes 1 + \sum_{n\geq 1} \frac{t^n}{(n)_{q^2}!} \left(\frac{k^{-1}_{\epsilon_1-\epsilon_{N+1}} - 1}{q^2-1}\right) \cdots \left(\frac{k^{-1}_{\epsilon_1-\epsilon_{N+1}} q^{2(n-1)} - 1}{q^2-1}\right) \otimes e^n_{\epsilon_1-\epsilon_{N+1}} = \mathcal{J}_1 \tag{21}$$

as one checks $\frac{k^{-1}_{\epsilon_1-\epsilon_{N+1}} q^{2(n-1)} - 1}{q^2-1} \in \hat{U}_q(\mathfrak{g})$. Applying the specialization map

$$a \mapsto \overline{a} := a \otimes_{\mathcal{A}} 1$$

to each of the tensor factors in $\mathcal{J}(e_{\epsilon_1-\epsilon_{N+1}})$ we come to a formula of [5, 10]:

$$\overline{\mathcal{J}(e_{\epsilon_1-\epsilon_{N+1}})} =$$

$$\exp(-t\textstyle\sum_{i=1}^{N-1} E_{1,i+1} \otimes E_{i+1,N+1} e^{-\frac{1}{2}\sigma_{1,N+1}}) \cdot$$

$$\left(1\otimes 1 + \sum_{n\geq 1} (-1)^n t^n \frac{H_{1,N+1}(H_{1,N+1}-1)\cdots(H_{1,N+1}-n+1)}{2^n n!} \otimes E_{1,N+1}\right)$$

where

$$\sigma_{1,N+1} = \ln(1 - t\ E_{1,N+1}) = -\sum_{n\geq 1} \frac{t^n}{n}\ E_{1,N+1}$$

and

$$E_{i,j} = \overline{e_{\epsilon_i-\epsilon_j}} \qquad H_{1,N} = \overline{\tfrac{k_{\epsilon_1-\epsilon_{N+1}}-1}{q-1}}.$$

4 The Cremmer-Gervais twist and its specialization at $q \to 1$

In this section we consider nontrivial quantum twists in $U_q(\mathfrak{sl}_3)[[t]]$ and their semi-classical limits $q \to 1$. As is known from the classification of Belavin-Drinfeld triples, there are two possible Belavin-Drinfeld triples for $\mathfrak{sl}_3$. The first one, the empty triple, is accounted for the Drinfeld-Jimbo deformation itself, while the second is associated with another deformation which can be called the Cremmer-Gervais quantization [2] and there is a solution to (1) defining the twisting element providing a possibility to deform $U_h(\mathfrak{sl}_3)[[h]]$ further. To be self-consistent we first recall the construction of this twist from [11] and then study different possibilities to define specialization $q \to 1$, unveiling a surprising connection with the Connes-Moscovici algebra $\mathcal{H}_1$.

Proposition 5 ([11]). *An element*

$$\mathcal{J}_{CG} = \Phi_{CG} \cdot \mathcal{K} = \exp_{q_h^{-2}}(\xi\ e_{32} \otimes e_{12})\ q_h^{h_{w_2} \otimes h_{w_1}}, \qquad \xi \in h \cdot \mathbb{C}[[h]],$$

where

$$h_{w_1} = \tfrac{2}{3} e_{11} - \tfrac{1}{3}(e_{22} + e_{33}), \qquad h_{w_2} = \tfrac{1}{3}(e_{11} + e_{22}) - \tfrac{2}{3} e_{33} \tag{22}$$

with $w_{1,2}$ being the fundamental weights, is a twist.

Proof. It is clear that $\mathcal{K} = q_h^{h_{w_2} \otimes h_{w_1}}$ defines an abelian twist of $U_{q_h}(\mathfrak{sl}_3)[[h]]$. It leads to the following new Hopf algebra $U_{q_h}^{\mathcal{K}}(\mathfrak{sl}_3)[[h]]$ with the same algebra structure as for $U_{q_h}(\mathfrak{sl}_3)[[h]]$ and the new deformed coproducts:

$$\Delta_{\mathcal{K}}(e_{12}) = q_h^{-2\ h_{1,-1}} \otimes e_{12} + e_{12} \otimes 1, \qquad \Delta_{\mathcal{K}}(e_{23}) = q_h^{h_{1,-2}} \otimes e_{23} + e_{23} \otimes q_h^{h_{1,0}}$$

$$\Delta_{\mathcal{K}}(e_{21}) = e_{21} \otimes q_h^{h_{2,-1}} + q^{-h_{0,1}} \otimes e_{21}, \qquad \Delta_{\mathcal{K}}(e_{32}) = e_{32} \otimes q_h^{-2h_{1,-1}} + 1 \otimes e_{32}$$

where $h_{m,n} := m\ h_{w_1} + n\ h_{w_2}$. We are done if we prove that Φ_{CG} is a twist for $U_{q_h}(\mathfrak{sl}_3)[[h]]$. Indeed, explicitly (1) reads as the following

$$\exp_{q_h^{-2}}(\xi\ e_{32} \otimes e_{12} \otimes 1) \cdot \exp_{q_h^{-2}}(\xi\ (e_{32} \otimes q_h^{-2h_{1,-1}} + 1 \otimes e_{32}) \otimes e_{12}) =$$

$$\exp_{q_h^{-2}}(\xi\ 1 \otimes e_{32} \otimes e_{12}) \cdot \exp_{q_h^{-2}}(\xi\ e_{32} \otimes (q_h^{-2\ h_{1,-1}} \otimes e_{12} + e_{12} \otimes 1))$$

and by the characteristic property of $q-$exponent

$$\exp_q(x+y) = \exp_q(y) \exp_q(x); \qquad xy = q\ yx$$

(1) holds. $\square$

Consider now the problem of defining specialization of $\mathcal{J}_{CG}$. To do so, we introduce from the beginning a $\mathbb{C}(q)$ analogue of $U_{q_h}^{\mathcal{K}}(\mathfrak{sl}_3)[[h]]$, which we denote by $U'_q(\mathfrak{sl}_3)$. As an algebra $U'_q(\mathfrak{sl}_3)$ is an extension of $U_q(\mathfrak{sl}_3)$ obtained by

attaching elements L_i (the maximal lattice), so that $K_j = \prod_{i=1}^{3} L_i^{a_{ij}}$. On the other hand, as a coalgebra $U_q'(\mathfrak{sl}_3)$ has a new coproduct fixed uniquely by its values on the Chevalley generators

$$\Delta(L_i) = L_i \otimes L_i$$

$$\Delta(e_1) = L_1^{-2}L_2^2 \otimes e_1 + e_1 \otimes 1, \qquad \Delta(e_2) = L_1 L_2^{-2} \otimes e_2 + e_2 \otimes L_1$$

$$\Delta(f_1) = f_1 \otimes L_1^2 L_2^{-1} + L_2^{-1} \otimes f_1, \qquad \Delta(f_2) = f_2 \otimes L_1^{-2}L_2^2 + 1 \otimes f_2,$$

where L_i are invertible, pairwise commuting and satisfying

$$L_i e_j L_i^{-1} = q^{\delta_{i,j}} e_j, \qquad L_i f_j L_i^{-1} = q^{-\delta_{i,j}} f_j.$$

In the sense of [?] define the regular form $\hat{U}_q'(\mathfrak{sl}_3)$ as a $\mathcal{A} := \mathbb{C}[q, q^{-1}]_{(q-1)}$ algebra such that there is an isomorphism

$$\hat{U}_q'(\mathfrak{sl}_3) \otimes_{\mathcal{A}} \mathbb{C}(q) \approx U_q'(\mathfrak{sl}_3).$$

$\hat{U}_q'(\mathfrak{sl}_3)$ is generated over $\mathcal{A}$ by the following set of generators

$$\left\{ L_i^{-1}, \frac{L_i - 1}{q-1}, e_i, f_i \right\}$$

If we denote by the "barred" generators the images of generators under specialization map

$$a \mapsto \overline{a} := a \hat{\otimes}_{\mathcal{A}} 1,$$

then the generators of $\hat{U}_q'(\mathfrak{sl}_3)$ specialize to the classical Chevalley generators of $U(\mathfrak{sl}_3)$ as the following:

$$\begin{array}{lll} \overline{e}_1 = E_{12}, & \overline{e}_2 = E_{23}, & \overline{\dfrac{L_1 - 1}{q-1}} = h_{w_1} \\ \overline{f}_1 = E_{21}, & \overline{f}_2 = E_{32}, & \overline{\dfrac{L_2 - 1}{q-1}} = h_{w_2} \end{array} \tag{23}$$

(see (22)). Let us additionally make completion $U_q'(\mathfrak{sl}_3)[[t]]$ then we can formulate a $q-$analogue of Proposition 5:

Proposition 6. *An element*

$$\hat{\Phi}_{CG} = \exp_{q^{-2}}(\zeta \ f_2 \otimes e_1), \qquad \zeta \in t \cdot \mathcal{A}[[t]]$$

is a twist in $\hat{U}_q'(\mathfrak{sl}_3)[[t]]$.

On the other hand apart from $\hat{\Phi}_{CG}$ we can construct the twists in $U_q'(\mathfrak{sl}_3)[[t]]$ which restrict to $\hat{U}_q'(\mathfrak{sl}_3)[[t]]$ only after a suitable change of basis which is implemented by some coboundary twist, namely we can prove

Proposition 7. *An element*

$$\mathcal{F}_{CG} = (V\otimes V)\exp_{q^{-2}}(\frac{q^{-2}\cdot\zeta\cdot t}{1-q^2}\ f_2\otimes e_1)\Delta(V^{-1}), \qquad \zeta\in t\cdot\mathcal{A}[[t]],$$

where

$$V = \exp_{q^{-2}}(-\frac{\zeta}{1-q^2}\ e_1L_1^2L_2^{-2})\cdot\exp_{q^2}(\frac{t}{1-q^2}\ f_2),$$

restricts to a twist of $\hat{U}_q'(\mathfrak{sl}_3)[[t]]$.

Proof. By the form of the coproducts

$$\Delta(e_1L_1^2L_2^{-2}) = e_1L_1^2L_2^{-2}\otimes L_1^2L_2^{-2} + 1\otimes e_1L_1^2L_2^{-2}, \qquad \Delta(f_2) = f_2\otimes L_1^{-2}L_2^2 + 1\otimes f_2$$

we have explicitly

$$\mathcal{F}_{CG} = \left(V\otimes\exp_{q^{-2}}(-\frac{\zeta}{1-q^2}\ e_1L_1^2L_2^{-2})\right)$$
$$\exp_{q^{-2}}(\frac{q^{-2}\cdot\zeta\cdot t}{1-q^2}\ f_2\otimes e_1)\exp_{q^{-2}}(-\frac{t}{1-q^2}\ f_2\otimes L_1^{-2}L_2^2)\Delta(\exp_{q^2}(\frac{\zeta}{1-q^2}\ e_1L_1^2L_2^{-2})).$$

Using the five terms relation, [4]:
If $[u,[u,v]]_{q^2} = [v,[u,v]]_{q^{-2}} = 0$ then

$$\mathrm{e}_{q^2}(u)\cdot\mathrm{e}_{q^2}(v) = \mathrm{e}_{q^2}(v)\cdot\mathrm{e}_{q^2}(\frac{1}{1-q^2}\,[u,v])\cdot\mathrm{e}_{q^2}(u) \tag{24}$$

we can simplify

$$\exp_{q^{-2}}(-\frac{t}{1-q^2}\ f_2\otimes L_1^{-2}L_2^2)\exp_{q^{-2}}(-\frac{\zeta}{1-q^2}\ 1\otimes e_1L_1^2L_2^{-2}) =$$
$$\exp_{q^{-2}}(-\frac{\zeta}{1-q^2}\ 1\otimes e_1L_1^2L_2^{-2})\exp_{q^{-2}}(\frac{q^{-2}\cdot\zeta\cdot t}{1-q^2}\ f_2\otimes e_1)\exp_{q^{-2}}(-\frac{t}{1-q^2}\ f_2\otimes L_1^{-2}L_2^2)$$

and thus $\mathcal{F}_{CG}$ transforms to the following form

$$\mathcal{F}_{CG} = \exp_{q^{-2}}(-\frac{t}{1-q^2}\,(f_2\otimes L_1^{-2}L_2^2+e_1L_1^2L_2^{-2}\otimes 1))\exp_{q^2}(\frac{t}{1-q^2}\,(f_2\otimes 1+L_1^2L_2^{-2}e_1\otimes L_1^2L_2^{-2}))$$

and finally by the Heine's formula we have

$$\mathcal{F}_{CG} = 1\otimes 1+\sum_{n\geq 1}\frac{1}{(n)_{q^2}!}\left(1\otimes\frac{L_1^{-2}L_2^2-1}{q^2-1}\cdots\frac{L_1^{-2}L_2^2q^{2(n-1)}-1}{q^2-1}\right)(t\cdot f_2\otimes 1+\zeta\cdot L_1^2L_2^{-2}e_1\otimes L_1^2L_2^{-2})^n$$

the latter expression restricts to $\hat{U}_q'(\mathfrak{sl}_3)[[t]]$ as we have

$$\frac{L_1^{-2}L_2^2q^{2(n-1)}-1}{q^2-1}\in\hat{U}_q'(\mathfrak{sl}_3)$$

$\square$

5 Semi-classical twists and noncommutative geometry

Proposition 8. *There is a semi-classical twist $\mathcal{F}$ and a homomorphism*

$$\iota : \mathcal{H}_1 \to U^{\mathcal{F}}(\mathfrak{sl}_3)[[t]]$$

Proof. We solve more general problem of obtaining quantization $\mathcal{H}'_{1,q}$ in the sense that there is an embedding

$$\iota_q : \mathcal{H}'_{1,q} \hookrightarrow U'_q(\mathfrak{sl}_3)[[t]]$$

where $\mathcal{H}'_{1,q}$ denotes an appropriately defined $q-$deformation of $\mathcal{H}'_1$. Consider the following coboundary twist in $U'_q(\mathfrak{sl}_3)[[t]]$:

$$\mathcal{J} := (W \otimes W)\Delta(W^{-1}), \qquad W = \exp_{q^{-2}}(-\frac{t}{1-q^2}\ e_{1+2}L_1)$$

where

$$e_{1+2}L_1 := (e_1e_2 - q\ e_2e_1)L_1$$

has the following coproduct

$$\Delta(e_{1+2}L_1) = e_{1+2}L_1 \otimes L_1^2 + 1 \otimes e_{1+2}L_1 + (1-q^2)\ e_1L_1^2L_2^{-2} \otimes e_2L_1.$$

By the properties

$$(1 \otimes e_{1+2}L_1)(e_{1+2}L_1 \otimes L_1^2 + (1-q^2)\ e_1L_1^2L_2^{-2} \otimes e_2L_1) =$$
$$q^{-2}(e_{1+2}L_1 \otimes L_1^2 + (1-q^2)e_1L_1^2L_2^{-2} \otimes e_2L_1)(1 \otimes e_{1+2}L_1),$$
$$(e_1L_1^2L_2^{-2} \otimes e_2L_1)(e_{1+2}L_1 \otimes L_1^2) = q^{-2}(e_{1+2}L_1 \otimes L_1^2)(e_1L_1^2L_2^{-2} \otimes e_2L_1)$$

$\mathcal{J}$ is nonsingular, the reasoning is same as in Proposition 4, and can be represented in the following form

$$\mathcal{J} = \overbrace{\mathrm{Ad}\exp_{q^{-2}}(-\frac{t}{1-q^2}\ e_{1+2}L_1 \otimes 1)(\exp_{q^2}(t\ e_1L_1^2L_2^{-2} \otimes e_2L_1))}^{\mathcal{J}_1} \cdot$$
$$\overbrace{\exp_{q^{-2}}(-\frac{t}{1-q^2}\ e_{1+2}L_1 \otimes 1)\exp_{q^2}(\frac{t}{1-q^2}\ e_{1+2}L_1 \otimes L_1^2)}^{\mathcal{J}_2}.$$

Let us check that for both factors we have

$$\mathcal{J}_{1,2} \in \hat{U}'_q(\mathfrak{sl}_3) \otimes \hat{U}'_q(\mathfrak{sl}_3)[[t]].$$

Indeed, it follows from explicit form of the factors $\mathcal{J}_{1,2}$:

$$\mathcal{J}_1 = \exp_{q^2}\left(t\ e_1L_1^2L_2^{-2}\ \frac{1}{1-t\ e_{1+2}L_1} \otimes e_2L_1\right),$$

$$\mathcal{J}_2 = \exp_{q^{-2}}\left(\frac{q^{-2}\ t}{1-q^{-2}}\ e_{1+2}L_1\otimes 1\right)\left(\exp_{q^{-2}}\left(\frac{q^{-2}\ t}{1-q^{-2}}\ e_{1+2}L_1\otimes L_1^2\right)\right)^{-1} =$$
$$= 1\otimes 1+\sum_{n\geq 1} t^n\frac{(-1)^n}{(n)_{q^{-2}}!}(e_{1+2}L_1)^n\otimes\frac{(L_1^2-1)}{q^2-1}\cdot\frac{(L_1^2q^{-2}-1)}{q^2-1}\cdots\frac{(L_1^2q^{-2(n-1)}-1)}{q^2-1}.$$

Calculating specialization at $q=1$ we come to

$$\overline{\mathcal{J}} = \exp(t\ E_{12}\frac{1}{1-t\ E_{13}}\otimes E_{23})\left(1\otimes 1+\sum_{n\geq 1}(-1)^n t^n E_{13}^n\otimes\frac{h_{w_1}(h_{w_1}-1)\cdots(h_{w_1}-n+1)}{n!}\right)$$

$\overline{\mathcal{J}}$ defines a noncoboundary deformation of $U(\mathfrak{sl}_3)[[t]]$ as it follows from

$$\overline{\mathcal{J}}_{21}\neq\overline{\mathcal{J}}.$$

On the other hand its quantum counter part $\mathcal{J}$ is coboundary in $U'_q(\mathfrak{sl}_3)[[t]]$ and amounts to switching to another basis of $U'_q(\mathfrak{sl}_3)[[t]]$. In particular, the subset of generators

$$\{L_1L_2^{-1}, e_1, f_2\}$$

is changing in the following way

$$L_1^2L_2^{-2}\mapsto W(L_1^2L_2^{-2})W^{-1} = L_1^2L_2^{-2},\qquad e_1\mapsto W(e_1)W^{-1} = e_1\frac{1}{1-t\ e_{1+2}L_1}$$

$$f_2\mapsto W(f_2)W^{-1} = f_2-\frac{t}{1-q^2}\ q^{-1}L_1^2L_2^{-2}e_1\ \frac{1}{1-t\ e_{1+2}L_1}$$

and we can form a Hopf subalgebra $\mathcal{D}_q\subset U_q'^{\mathcal{J}}(\mathfrak{sl}_3)[[t]]$ generated by the set of generators

$$\left\{k := L_1^2L_2^{-2}, x := f_2, z := q^{-1}\ L_1^2L_2^{-2}e_1\frac{1}{1-t\ e_{1+2}L_1}\right\}.$$

with the following structure:

$$kxk^{-1} = q^2\ x,\qquad kzk^{-1} = q^2\ z,\qquad q^2xz-zx = -tz^2$$

$$\Delta(k) = k\otimes k,\qquad \Delta(z) = z\otimes k+1\otimes z$$
$$\Delta(x) = x\otimes k^{-1}+1\otimes x+t\ z\otimes\frac{(k-k^{-1})}{1-q^2}.$$

The structure of its specialization $\mathcal{D}_1\approx\mathcal{D}_q\hat{\otimes}_{\mathcal{A}}1$ is the following:

$$[\overline{y},\overline{x}] = \overline{x},\qquad [\overline{y},\overline{z}] = \overline{z},\qquad \overline{xz}-\overline{zx} = -t\overline{z}^2$$

$$\Delta(\overline{y}) = \overline{y}\otimes 1+1\otimes\overline{y},\qquad \Delta(\overline{z}) = \overline{z}\otimes 1+1\otimes\overline{z}$$

$$\Delta(\overline{x}) = \overline{x} \otimes 1 + 1 \otimes \overline{x} - t\ \overline{z} \otimes \overline{y}$$

where

$$\overline{y} := \overline{\frac{k-1}{q-1}}.$$

Finally we obtain the stated map

$$\iota : \mathcal{H}_1' \to \mathcal{D}_1 \subset U^{\overline{\mathcal{J}}}(\mathfrak{sl}_3)[[t]]$$

by fixing its values on the generators

$$\iota(X) = -\tfrac{1}{2}\overline{x}, \qquad \iota(Y) = \overline{y}, \qquad \iota(Z) = t\overline{z}.$$

where the generators X, Y, Z fulfills the relations

$$[Y, X] = X, \qquad [Y, Z] = Z, \qquad [X, Z] = \tfrac{1}{2}Z^2$$

and the coproducts are obtained from (2) by substitution Z for δ_1. Next, ι is an isomorphism as ι maps the basis of $\mathcal{H}_1' : \left\{X^k Y^l Z^m\right\}_{k,l,m\geq 0}$ onto the basis of $\mathcal{D}_1 : \left\{\overline{x}^k \overline{y}^l \overline{z}^m\right\}_{k,l,m\geq 0}$ □

Now we can obtain

Proposition 9. *An element*

$$F_q = (WV \otimes WV) \exp_{q^{-2}}\left(\frac{q^{-3} \cdot t^3}{(1-q^2)^2}\ f_2 \otimes e_1\right)\Delta(V^{-1})(W^{-1} \otimes W^{-1}),$$

where

$$V = \exp_{q^{-2}}\left(-\frac{q^{-1}t}{(1-q^2)^2}\ e_1 L_1^2 L_2^{-2}\right) \cdot \exp_{q^2}\left(\frac{t}{1-q^2}\ f_2\right),$$

restricts to a twist in $\hat{U}_q'^{\mathcal{J}}(\mathfrak{sl}_3)[[t]]$.

Proof. It is convenient to introduce notation

$$\tilde{\mathcal{F}}_{CG} = (V \otimes V) \exp_{q^{-2}}\left(\frac{q^{-3} \cdot t^3}{(1-q^2)^2}\ f_2 \otimes e_1\right)\Delta(V^{-1}).$$

Then by the reasoning of Proposition 6 we can prove that

$$\tilde{\mathcal{F}}_{CG} = 1\otimes 1 + \sum_{n\geq 1} \frac{1}{(n)_{q^2}!}\left(1 \otimes \frac{L_1^{-2}L_2^2 - 1}{q^2-1} \cdots \frac{L_1^{-2}L_2^2 q^{2(n-1)} - 1}{q^2-1}\right)\left(t \cdot f_2 \otimes 1 + \frac{q^{-1}t^2}{1-q^2} \cdot L_1^2 L_2^{-2} e_1 \otimes L_1^2 L_2^{-2}\right)^n$$

Next, the following element

$$\tilde{\mathcal{F}}_{CG}^W := (W \otimes W)\tilde{\mathcal{F}}_{CG}\Delta(W^{-1})$$

is a twist in $U_q'(\mathfrak{sl}_3)[[t]]$ and respectively

$$F_q = (W \otimes W)\tilde{\mathcal{F}}_{CG}(W^{-1} \otimes W^{-1})$$

defines a twist in $U_q'^{\mathcal{J}}(\mathfrak{sl}_3)[[t]]$. Explicitly

$$F_q = 1\otimes 1 + \sum_{n\geq 1} \frac{1}{(n)_{q^2}!}\left(1\otimes \frac{{L_1^{-2}}L_2^2-1}{q^2-1}\cdots\frac{{L_1^{-2}}L_2^2 q^{2(n-1)}-1}{q^2-1}\right)\cdot$$
$$\left(t\cdot f_2\otimes 1 - t^2\cdot q^{-1}\ e_1\frac{1}{1-t\ e_{1+2}L_1}\otimes\frac{L_1^2{L_2^{-2}}-1}{q^2-1}\right)^n$$

again similarly to Proposition 6 we check that F_q restricts to a twist in $\hat{U}_q'^{\mathcal{J}}(\mathfrak{sl}_3)[[t]]$. If we specialize $q=1$ and apply Proposition 8 we obtain

$$F_1 = 1\otimes 1 + \sum_{n\geq 1}\frac{t^n}{n!}(1\otimes Y(Y-1)\cdots(Y-n+1))(2X\otimes 1 + Z\otimes Y)^n.$$

Note that a formula for F_1 was obtained in [7] by a direct check in the study of the twists for $\mathfrak{sl}_2$ Yangian. To make correspondence with (3) we additionally twist F_1 by a coboundary:

$$F_1' = (\exp(t\ XY)\otimes\exp(t\ XY))\,F_1\Delta(\exp(-t\ XY))$$

expanding in t we have

$$F_1' = 1\otimes 1 + t\cdot(X\otimes Y - Y\otimes X + ZY\otimes Y)+\cdots$$

Thus F_1' is equivalent to (3). $\square$

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